Finite-Difference Solution to the 2-D Heat Equation

MSE 350

Problem Overview

Given:

- Initial temperature in a 2-D plate
- Boundary conditions along the boundaries of the plate.

Find: Temperature in the plate as a function of time and position.

Energy equation:

$$\rho C_p \frac{\partial T}{\partial t} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

$$T(x, 0, t) = \text{given}$$

$$T(x, H, t) = \text{given}$$

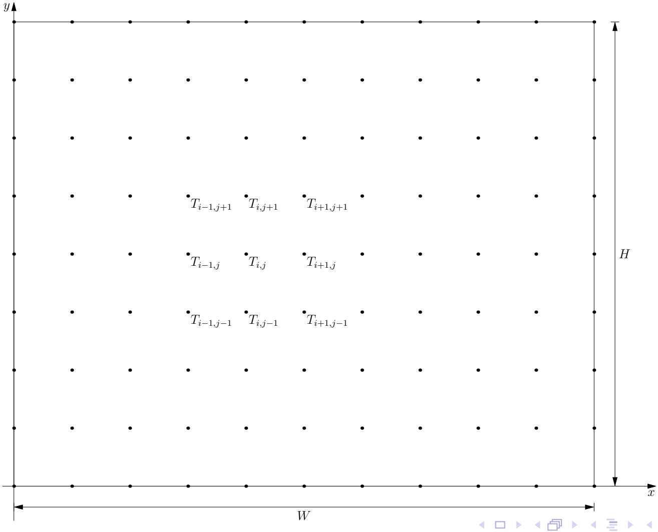
$$T(0, y, t) = \text{given}$$

$$T(W, y, t) = \text{given}$$

$$T(x, y, 0) = \text{given}$$

Solution Overview

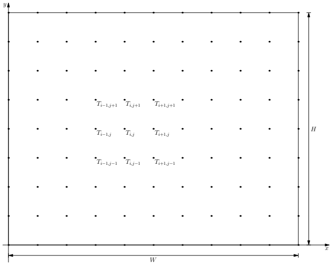
Approach: *discretize* the temperatures in the plate, and convert the heat equation to finite-difference form.



$$T_{i,j}^{k}$$

$$i, j = \text{location (node numbers)}$$

$$k = \text{time (time step number)}$$



$\Delta x, \Delta y =$ node spacings in the x and y directions.

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

$$\frac{T_{i,j}^{k+1} - T_{i,j}^{k}}{\Delta t} = \alpha \left[\left(\frac{T_{i-1,j}^{k} - 2T_{i,j}^{k} + T_{i+1,j}^{k}}{\Delta x^2} \right) + \left(\frac{T_{i,j-1}^{k} - 2T_{i,j}^{k} + T_{i,j+1}^{k}}{\Delta y^2} \right) \right]$$

Discretizing the Heat Equation (Explicit)

If $\Delta x = \Delta y = h$:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

$$\frac{T_{i,j}^{k+1} - T_{i,j}^{k}}{\Delta t} = \alpha \left(\frac{T_{i,j-1}^{k} + T_{i-1,j}^{k} - 4T_{i,j}^{k} + T_{i+1,j}^{k} + T_{i,j+1}^{k}}{h^2} \right)$$

If $\Delta x = \Delta y = h$:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

$$T_{i,j}^{k+1} = T_{i,j}^{k} + \Delta t \alpha \left(\frac{T_{i,j-1}^{k} + T_{i-1,j}^{k} - 4T_{i,j}^{k} + T_{i+1,j}^{k} + T_{i,j+1}^{k}}{h^2} \right)$$

Stability

If $\Delta x = \Delta y = h$:

$$T_{i,j}^{k+1} = \left(1 - \frac{4\Delta t\alpha}{h^2}\right) T_{i,j}^{k} + \Delta t\alpha \left(\frac{T_{i,j-1}^{k} + T_{i-1,j}^{k} + T_{i+1,j}^{k} + T_{i,j+1}^{k}}{h^2}\right)$$

Coefficient on $T_{i,j}^{k}$ must be non-negative for stability.

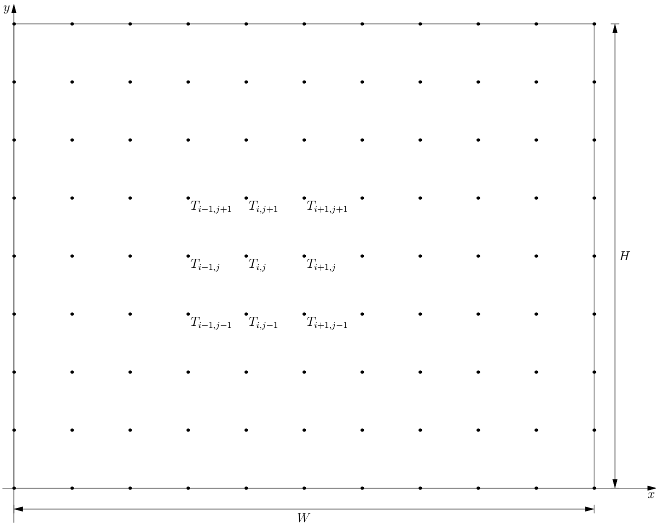
Hence,

$$\left(1 - \frac{4\Delta t\alpha}{h^2}\right) \geq 0$$

so

$$\Delta t \leq \frac{h^2}{4\alpha}$$

BOUNDARIES



What do we do about the edges? Same as in a 1-D bar.

BOUNDARIES

What do we do about the edges?

- If we know the temperature of the boundaries already, we don't need to write equations for those nodes.
- If we know the temperature derivitive there, we invent a *phantom node* such that $\frac{\partial T}{\partial x}$ or $\frac{\partial T}{\partial y}$ at the edge is the prescribed value.

At steady-state, time derivatives are zero:

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = 0$$

$$\left[\left(\frac{T^k_{i-1,j} - 2T^k_{i,j} + T^k_{i+1,j}}{\Delta x^2}\right) + \left(\frac{T^k_{i,j-1} - 2T^k_{i,j} + T^k_{i,j+1}}{\Delta y^2}\right)\right] = 0$$

Same Δx and Δy ($\equiv h$):

$$T_{i,j} = \frac{1}{4}\left(T_{i,j-1} + T_{i-1,j} + T_{i+1,j} + T_{i,j+1}\right)$$